Tensors with Penrose Diagrammatic Notation

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Contents

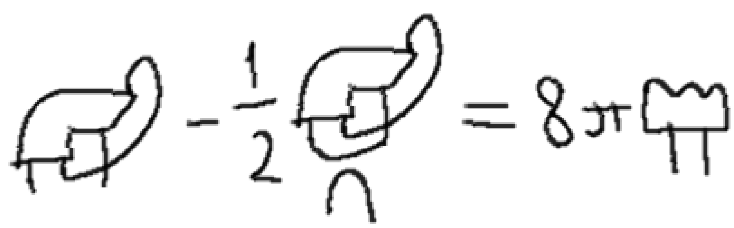


Figure 1: **Einstein field equations in natural units with zero cosmological constant with the diagrammatic notation**

1 Review of tensor algebra

- A rank (p, q) tensor is an element of a tensor product space $\overbrace{\mathrm{V}^* \otimes \cdots \mathrm{V}^*}^{\text{p-times}} \otimes \overbrace{\mathrm{V} \otimes \cdots \mathrm{V}}^{\text{q-times}}$, or a multilinear map which feeds on p vectors and q dual vectors to give out a number.

- Examples:
  •Any scalar is a rank $(0, 0)$ tensor.

  • $v(\overbrace{\ \_\_\ }^{\in \mathrm{V}^*}) \in \mathrm{V}$ is a rank $(1, 0)$ tensor (or a rank 1 tensor).

  It has one slot for a dual vector, so for $w \in V^*$, $v(w)$ is a scalar. In matrices representation, we know this evaluation is just multiplication of the vector v by the row vector w from the left; wv.

- $w(\overbrace{\ \_\ }^{\in \mathrm{V}}) \in \mathrm{V}^*$ is a rank $(0,1)$ tensor (or a rank 1 tensor).

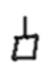
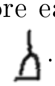
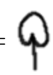
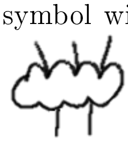
It has one slot for a dual vector, so for $v \in V$, $w(v)$ is a scalar. In matrices representation, we know this is also wv. So we see that a dual vector acting a vector $w(v)$ is the same as the vector acting on the dual $v(w)$.

- $A(\overbrace{\ \_\ }^{\in \mathrm{V}^*}, \overbrace{\ \_\ }^{\in \mathrm{V}}) \in \mathrm{V} \otimes \mathrm{V}^*$ is a rank 2 tensor

Filling the first slot $A(v, \_)$ with $v \in \mathrm{V}$ makes it a rank $(0,1)$ tensor (a dual vector). In matrix representation, it's equivalent to multiplying the column vector v from the right; the result of Av is indeed a column vector. Filling the second slot $A(\_, w)$ with $w \in V^*$ makes it a rank $(1,0)$ tensor (a vector). In matrix represention, it's multiplying the row vector w from the left of the matrix; wA. Filling both slots $A(v, w)$ give a rank $(0,0)$ tensor which is a scalar. In matrix representation, it's equivalent to multiplying the matrix from the left and the right wAv, and this indeed gives a scalar.

- Alternatively, one can fill two slots with one rank-2 tensor. Or three slots with one rank-1 tensor and one rank-2 tensor, and so forth.

2 Penrose diagrammatic notation

- A rank (p, q) tensor is drawn as a distinctive symbol with p lines pointing upwards and q lines pointing downwards. Examples:
  - $v, t, r \in \mathrm{V}$ are rank (1, 0) tensors (vectors) and therefore each is denoted by a symbol with one line projecting upwards: $v \mathrel{\hat{=}}$ , $t \mathrel{\hat{=}}$ , $r \mathrel{\hat{=}}$ .
  - $w, m, n \in \mathrm{V}^*$ are rank (0, 1) tensors (covectors) and therefore denoted by symbols with one line projecting downwards: $w \mathrel{\hat{=}}$ , $m \mathrel{\hat{=}}$ , $n \mathrel{\hat{=}}$ .
  - $H \in \mathrm{V}^* \otimes \mathrm{V}^*$ is a rank (0, 2) tensor and therefore denoted by a symbol with two lines projecting downwards: $H \mathrel{\hat{=}}$ .
  - $A, B \in \mathrm{V} \otimes \mathrm{V}^*$ are rank (1, 1) tensors (matrices) and therefore denoted by symbols with one line projecting upwards and one line projecting downwards: $A \mathrel{\hat{=}}$ , $B \mathrel{\hat{=}}$ .
  - $Q \in \mathrm{V} \otimes \mathrm{V}^* \otimes \mathrm{V} \otimes \mathrm{V}^* \otimes \mathrm{V}$ is a mixed rank (3, 2) tensor and therefore denoted by a symbol with three lines projecting upwards and two lines projecting downwards: $Q \mathrel{\hat{=}}$ . The order the slots appears horizontally is of importance in specifying the type of tensor when its mixed.

- Application/composition of tensors is represented by connecting the free lines of the symbols together:

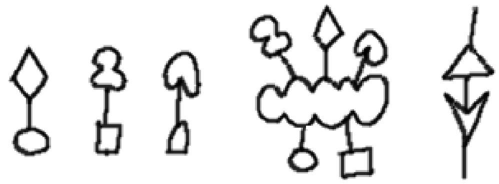
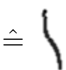
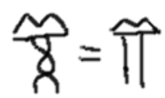


Figure 2: **We see in order:** $w(\underline{v})$, $m(\underline{t})$, $n(\underline{r})$, $Q(\underline{m}, \underline{v}, \underline{w}, \underline{t}, \underline{n})$ **and the matrix product** AB

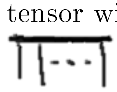
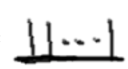
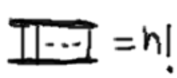
- It follows that scalars are objects without lines projecting upwards nor downwards. So as can be seen in the figure above, $w(v), m(t), n(r)$ and $Q(m, v, w, t, n)$ would all be scalars (rank (0, 0) tensors). The matrix product AB however has two lines projecting out of it still, so it's a rank (1, 1) tensor (product of two matrices is a matrix).

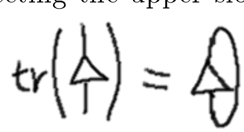
- The identity matrix is simply represented by a vertical line: $(\delta^{\mu}{}_{\nu} \hat{=})I \hat{=}$ . Indeed, it's a rank (1, 1) tensor and thus has one end projecting upwards and one end projecting downwards. Composing the identity with any slot should give back the same tensor. With this choice for the symbol for the identity map, composing it would be just be extending a line ending with a vertical line, which notationally has no meaning; i.e. tensor doesn't change.

- Swapping two slots of a tensor is represented by just extending the slot lines so they exchange the order in which they appear horinzontally. e.g. swapping the first and the third slot of the tensor Q yields: 

- Swapping two slots twice yields the same tensor again: 

- The antisymmetrisation operation between several slots is represented by drawing a thick-line through the slot lines. Whereas the symmetrisation oneartion between severeal slots is represented by drawing a wiggly line through the slot lines:

- The antisymmetric epsilon tensor with all lower slots is represented by a thick bar with lines projected downwards: $\varepsilon \hat{=}$ . The antisymmetric epsilon tensor with all upper slots is represented by a thick bar with lines projected upwards: $\epsilon \hat{=}$ . ε and ϵ are normalised by: 

- In einstein's notation, the partial trace over two slots is represented by equating the respective upper and lower indices (e.g. $A^{\mu}{}_{\mu}$); or equivalently contracting the indices with the identity ($A^{\mu}{}_{\nu}\delta^{\nu}{}_{\mu}$). In penrose notation, it is thus represented by connecting the upper slot with the lower. For example, the trace of a matrix A is represented by: .

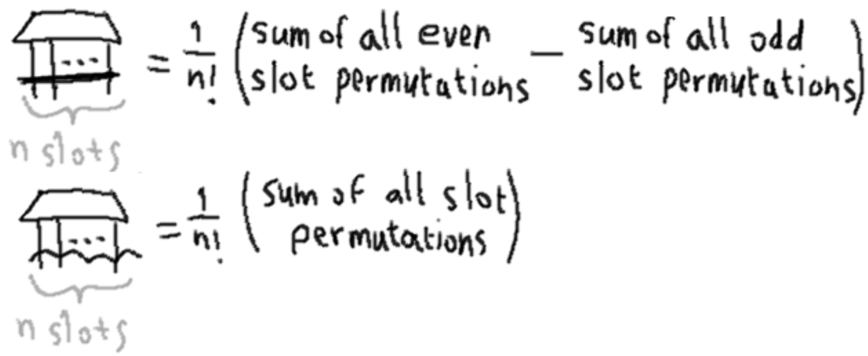


Figure 3: **The antisymmetrisation opeartion is the sum of all possible even permutations (achieved by an even number of swaps on the slots) minus all possible odd permutations (achieved by an odd number of swaps) multiplied by the total number of permutations $1/n!$ for normalisation. The symmetrisation operation is similar, except that all permutations, even and odd, are summed over.**

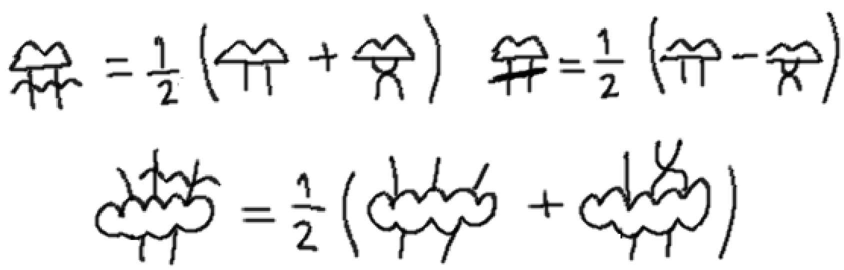


Figure 4: **Examples of explicitly writing out the symmetrisation and antisymmetrisation.**

Figure 5: **Any rank 2 tensor can be decomposed into a symmetric part and an antisymmetric part.**

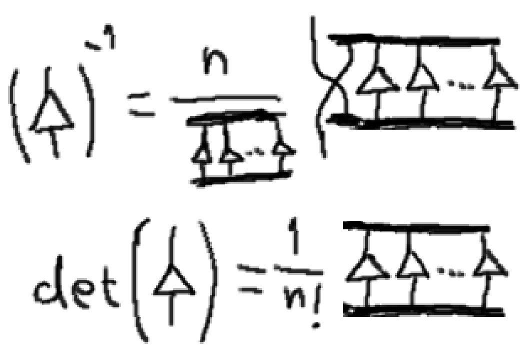
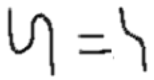


Figure 6: **The inverse and the determinant of an invertible matrix A written in the diagrammatic notation.**

- It is then convenient to represent the metric (0, 2) tensor with a hoop symbol $g \hat{=} \bigcap$ and the inverse metric (2, 0) tensor with an inverted hoop $g^{-1} \hat{=}$: . Joining the two hoops (composition of the metric with its inverse) then naturally give a vertical line (i.e. identity): .

- Composing the hoops with a matrix defines its transpose.

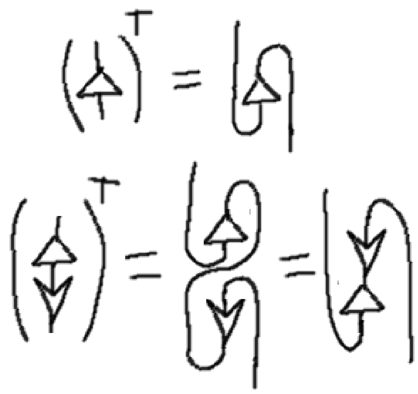
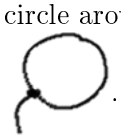
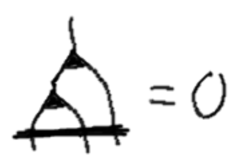


Figure 7: **The transpose of A is simply composing hoops (in einstein's notation: lower the upper index and raise the lower index). The second line demonstrates how** $(AB)^T = B^T A^T$

- The covariant derivative of a tensor is represented by a circle around the tensor's symbol with one lower slot line projecting down out of the circle: . For instance, the covariant derivative of Q (in einstein's notation $\nabla_\mu Q^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta}{}^{\nu}$) is represented as: 

- The structure coeffecients defined by $[e_a, e_b] = f_{ab}{}^c e_c$ are represented by $f_{ab}{}^c \hat{=}$ . The jacobi identity is then given by: 

3 Demonstrations

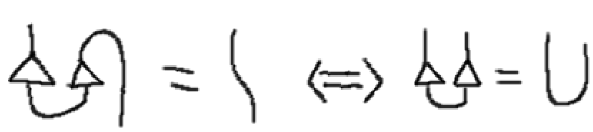


Figure 8: **The hermiticity/orthogonality condition** $AA^T = I$

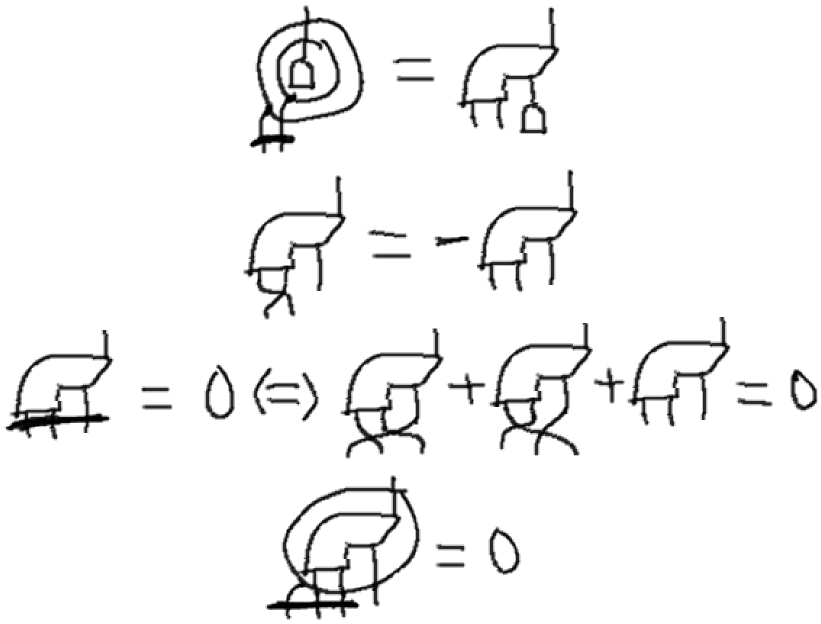


Figure 9: **The first expression is the Ricci identity defining the Riemann tensor $[\nabla_\mu, \nabla_\nu] r_\alpha = R_{\mu\nu\alpha}{}^{\beta} r_\beta$. The second expression is the antisymmetry of the Riemann tensor in its first two indices. The third expression is the 1st Bianchi identity $R_{[\mu\nu\alpha]}{}^{\beta} = 0$ and the fourth expression is the 2nd (or differential) Bianchi identity $\nabla_{[\beta} R_{\mu\nu]\alpha}{}^{\beta} = 0$.**

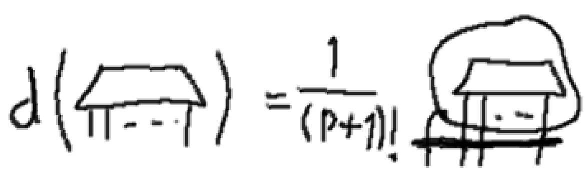


Figure 10: **Expression for the exterior derivative of a p-form**

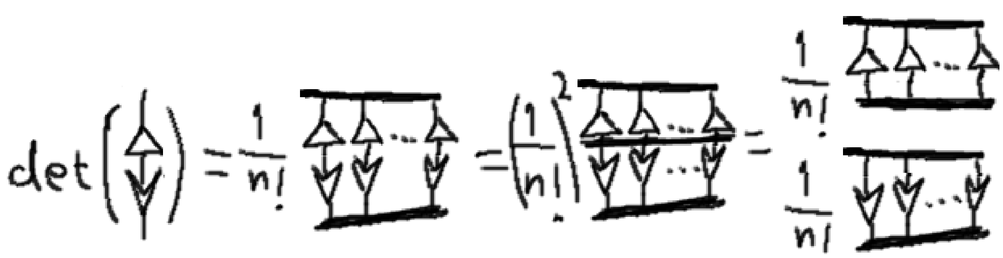


Figure 11: **Determinant of the product is the product of the determinants.**

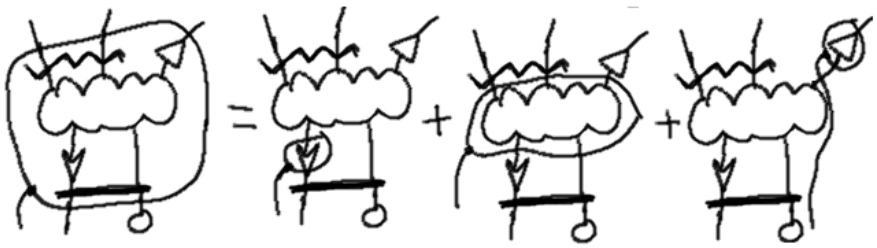


Figure 12: **Leibniz rule expansion** $\nabla_\mu(B^\beta{}_{[\kappa}Q^{(\alpha}{}_{|\beta|}{}^{\gamma)}{}_{\delta]}{}^\nu A^\eta{}_\nu)v^\delta = \nabla_\mu(B^\beta{}_{[\kappa})Q^{(\alpha}{}_{|\beta|}{}^{\gamma)}{}_{\delta]}{}^\nu A^\eta{}_\nu v^\delta + B^\beta{}_{[\kappa}\nabla_\mu(Q^{(\alpha}{}_{|\beta|}{}^{\gamma)}{}_{\delta]}{}^\nu)A^\eta{}_\nu v^\delta + B^\beta{}_{[\kappa}Q^{(\alpha}{}_{|\beta|}{}^{\gamma)}{}_{\delta]}{}^\nu)\nabla_\mu(A^\eta{}_\nu)v^\delta$

4 References

- Roger Penrose, "The Road to Reality"
- Wikipedia: `https://en.wikipedia.org/wiki/Penrose_graphical_notation`