GR Marathon - Light Jogging with Special Relativity

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1 Events, Inertial Frames of Reference and Galilean Relativity

- An *event* is defined a point $\vec{x}$ in space at a certain time t. We write them together $(t, \vec{x})$
- An *inertial frame of reference* is a non-accelerating frame of reference. Alternatively, it's a reference frame where Newton's laws holds. In particular, it's usually Newton's second law that gets violated in non-inertial frames of reference (acceleration appears without a force).

1.1 Example of an inertial frames of reference

- If you set up a lab on an airplane, Newton's second law will still hold; moving objects keep moving without accelerating and stationary objects remain stationary unless an external force intervenes (friction, pushing ..etc). So the frame of the lab is an inertial frame of reference. You can place a cup of water on your table and even play a game of pool with your friend.
- If you set up a lab on the surface of the earth (neglecting that the earth rotates), Newton's second law will also apply.

1.2 Non-example of inertial frames of reference

- If you set up your small lab on a roller coaster, then you attempt to place your pool balls stationary on the table, your game will quickly be interrupted everytime the roller coaster takes a turn or go through a loop. You'll also quickly find that cup of water has also moved and that the water is no longer in the cup. Stationary objects don't remain stationary despite you not applying any forces on them. This violates Newton's second law, and so this frame is not inertial.

1.3 Galilean transformations

- For the two inertial frames of reference: the earth's and the plane's. Is the plane's lab moving and the earth is stationary, or is the earth moving and the plane is stationary?
- If you have your axes such that the origin is always moving with the lab on the plane, you're doing physics *in the reference frame of the plane*
- Similarly, if you have your axes such that the they're always moving in the center of your lab on earth, then you're doing physics *in the reference frame of the earth*
- There's **no absolute inertial frame** or one that is special; all inertial frames are just as good.
- Suppose a ball is stationary in the plane's frame (with respect to the origin on the plane's lab) at position x_P and time t_P. What would be the event of the ball (t'_P, x'_P) at a later time t'_P?

$$\begin{aligned} x'_P &= x_P && \text{;ball is stationary, so new position equals old position} \\ t'_P &= t'_P && \text{;the new time is the new time} \end{aligned}$$

- What about in the reference frame of the earth? What would be the event of the ball at a later time t'_E if the plane is moving away from the lab on earth at velocity u?

$$\begin{aligned} x'_E &= x'_P + ut'_P && \text{;position evolves as ball is moving in this frame} \\ t'_E &= t'_P && \text{;time on the plane equals the time on earth later} \end{aligned}$$

- If the plane is stationary with respect to the lab on earth, $u = 0$ and:

$$\begin{aligned} x'_E &= x'_P + ut'_P = x'_P \\ t'_E &= t'_P \end{aligned}$$

which tells us that the two frames are exactly the same.

- These transformations from one reference frame to another are called *Galilean transformations.*
- We can see that the relative velocity between the two reference frames is what differentiates them and is what determines what the galilean transformation.
- We'll denote a galilean transformation from frame A to another frame B moving with velocity u relative to A as G_u.
- Using fancy vector notation, we could have written the transformations above from frame of the plane P to frame of the earth E with G_u being a matrix and the event as a vector:

$$\begin{pmatrix} t'_E \\ x'_E \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} t'_P \\ x'_P \end{pmatrix} = \begin{pmatrix} t'_E \\ ut'_P + x'_P \end{pmatrix}$$

so G_u can be expressed as a matrix:

$$G_u := \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$$

which you multiply the event vector by.

1.4 Three reference frames

- Suppose we have three inertial frames of reference. One reference frame T attached to a tree, another reference frame C attached to a canoe (or a boat), and one more reference frame B attached to a ball.
- The ball - and therefore the frame B attached to it - is moving with velocity v relative to the canoe - or to its frame C.
- The canoe with its frame C is moving with velocity u with repsect to the frame tree with its frame T.
- The ball is always stationary in the reference frame B since it's attached to it. Therefore in B, the event of the ball at a later time t'_B is:

$$x'_B = x_B$$
$$t'_B = t'_B$$

- What about in the reference frame of the canoe C? We know the velocity of the ball in this frame is v, so we just use a galilean transformation to get the evolved coordinates in the frame C, just like before:

$$
\begin{aligned}
x'_C &= x'_B + ut'_B \\
t'_C &= t'_B
\end{aligned}
$$

- What about in the reference frame of the tree T? We only know that the canoe is moving with velocity u relative to the tree, but we don't know yet how the ball is moving relative to the tree (you do know it'll be $u+v$, but that's what we're trying to prove). What would be (t'_T, x'_T)?

- Idea: We do a two-way galilean transformation

$$
(t'_B, x'_B) \xrightarrow{G_v} (t'_C, x'_C) \xrightarrow{G_u} (t'_T, x'_T)
$$

Let's compute that manually, and use the fancy vector notation:

$$
\begin{aligned}
\text{We know that:} \quad & \begin{pmatrix} t'_C \\ x'_C \end{pmatrix} = G_v \begin{pmatrix} t'_B \\ x'_B \end{pmatrix} \\
\text{and that:} \quad & \begin{pmatrix} t'_T \\ x'_T \end{pmatrix} = G_u \begin{pmatrix} t'_C \\ x'_C \end{pmatrix} \\
\text{substituting:} \quad \Longrightarrow \quad & \begin{pmatrix} t'_T \\ x'_T \end{pmatrix} = G_u G_v \begin{pmatrix} t'_B \\ x'_B \end{pmatrix} \\
& = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} t'_B \\ x'_B \end{pmatrix} \\
& \begin{pmatrix} t'_T \\ x'_T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u+v & 1 \end{pmatrix} \begin{pmatrix} t'_B \\ x'_B \end{pmatrix} = \begin{pmatrix} t'_B \\ (u+v)t'_B + x'_B \end{pmatrix}
\end{aligned}
$$

The main result here is that applying G_v to go from B to C then G_u to go from C to T is equivalent to applying G_{u+v} to go immediately from B to T. This tells us that *velocities add up*. This is called the **Galilean law addition of velocities**. It's something that we all know very well; that if you can pitch a ball at 100 mph, then the catcher would recieve it at 100 mph. But if you do it while on top of a skateboard moving at 10 mph, then the catcher will recieve a 100 mph + 10 mph = 110 mph fast ball.

However this will turn out not to be the case in great generality. In particular, with speeds near the speed of light, velocities don't add up like that.

2 Special Relativity

2.1 Motivation

- What if you shine light in reference frame B, and then view it in frame A that moving with velocity u w.r.t frame B? Galilean law of addition of velocities tells you that the velocity of light would be $v + c$.
- Problem: Maxwell's theory of electromagnetism suggests that light is an electromagnetic wave, and that all electromagnetic waves travel at speed $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$.
- Both the magnetic constant μ_0 and the electric constant ε_0 are constants that have the same value in all reference frames, and c should there just be $\frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ in all frames. Contradiction?
- People concluded that light, like any wave, must be travelling in a medium that they called 'ether'. Ether can be dragged around with a reference frame so Maxwell's theory still holds.
- Michelson-Morley would then in 1887 do an experiment involving mirrors and making use of interference properties and show that, contrary to what they would have liked, that light does not travel in a medium.
- So either Maxwell or Newton and his company did some mistake. Einstein sided with Maxwell.

2.2 Postulates of Special Relativity

Einstein set himself two postulates which his new theory should follow:

1. The laws of physics are the same in all inertial frames of reference.
2. The speed of light c is the same in all inertial frames of reference.

We can see that both postulates were taken one step further from what we've taken previously. We previously only wanted Newton's laws to be the same in all inertial frames, but now it's all the laws of physics. In the second postulate, Einstein did not only postulate that *electromagnetic waves* in particular travel in c in all reference frames; rather he took the speed c as special, and anything that travels at c must travel at c in all inertial frames.

2.3 The new transformation

- Before we start thinking of a new transformation that would keep c constant in all inertial frames, we must consider that for $\frac{v}{c} << 1$, this transformation must correspond to the galilean transformation. This is known as *the correspondence principle.*

- The plot of the position of object x against time t is called its *spacetime diagram.*

- It's clear that the spacetime diagram of an object moving at velocity u is a straight line, as long as there are no forces acting on it. This is just Newton's second law, as any accleration will result in a curve.

- For the Newton's second law to hold, it must mean that going from one reference frame to another should use a transformation that keeps the straight lines in spacetime diagrams straight, since any curve implies acceleration (change in slope is a change in velocity).

- Geometrically, a galilean transformation 'shears' (sort of "stretches") the spacetime diagram in the direction of x. Shearing can never make straight lines in spacetime diagrams curve.

- We also know that the new transformation we're looking for must also keep the straight lines straight for Newton's second law to hold in all reference frames. A change between inertial frames shouldn't introduce any acceleration.

- In other words, the transformation is linear. Any linear transformation can be represented by a matrix.

- Let's call denote this transformation matrix by L_u, where u is a velocity characterising it. The most general form of this matrix is:

$$L_u = \begin{pmatrix} \theta(u) & \tau(u) \\ \alpha(u) & \gamma(u) \end{pmatrix}$$

where $\theta(u)$, $\tau(u)$, $\alpha(u)$ and $\gamma(u)$ are unknown functions of u. What we'll do next is find these functions, and therefore determine the matrix L_u.

2.4 Finding the Lorentz transformation

- Letting go of any previously held notions of galilean transformations of how velocities add up, the following are things we least expect from this new transformation L_u:

$$L_u \begin{pmatrix} t \\ -ut \end{pmatrix} \overset{!}{=} \begin{pmatrix} t' \\ 0 \end{pmatrix}$$; in a frame where your frame is moving at u, an object that is moving at $-u$ in your frame should look stationary

$$L_{-u} L_u = L_u L_{-u} \overset{!}{=} I$$; this should be identity for $u \neq c$

$$L_u \begin{pmatrix} t \\ ct \end{pmatrix} \overset{!}{=} \begin{pmatrix} t' \\ ct' \end{pmatrix}$$; the second postulate: the speed of light c is the same in any frame

$$L_0 \overset{!}{=} I$$; going from a frame to another comoving frame must be identity

$$L_c L_u \neq L_{c+u}$$; velocities should not add up in the galilean sense

- Only the first three conditions are needed to find L_u completely. The last two will just be used to test that we have arrived at the correct result.

- Note that with galilean transformations, time never transformed; it would always be the same no matter what reference frame we pick. However, here, we do not exclude that the time component gets transformed. And indeed, as we shall see, t is not always equal to t'.

2.4.1 Finding $\alpha(u)$ in terms of $\gamma(u)$ and showing that $\gamma(u) = \gamma(-u)$:

- Starting from the first condition:

$$\begin{pmatrix} \theta(u) & \tau(u) \\ \alpha(u) & \gamma(u) \end{pmatrix} \begin{pmatrix} t \\ -ut \end{pmatrix} = \begin{pmatrix} t\theta(u) - ut\tau(u) \\ t\alpha(u) - ut\gamma(u) \end{pmatrix} p \overset{!}{=} \begin{pmatrix} t' \\ 0 \end{pmatrix}$$

$$t\alpha(u) - ut\gamma(u) = 0 \implies \boxed{\alpha(u) = u\gamma(u)}$$

- Let's update the matrix L_u so that it no longer has $\alpha(u)$:

$$L_u = \begin{pmatrix} \theta(u) & \tau(u) \\ u\gamma(u) & \gamma(u) \end{pmatrix}$$

- The second condition is that the inverse of all of our transformations L_u must be L_{-u}. Let's try to hit the expression of the first condition with L_{-u} on both sides to see if we can squeeze out any more info from it:

$$\cancel{(L_{-u})L_u}\begin{pmatrix} t \\ -ut \end{pmatrix} \stackrel{!}{=} (L_{-u})\begin{pmatrix} t' \\ 0 \end{pmatrix} = \begin{pmatrix} \theta(-u) & \tau(-u) \\ -u\gamma(-u) & \gamma(-u) \end{pmatrix}\begin{pmatrix} t' \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} t \\ -ut \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} t'\theta(-u) \\ -ut'\gamma(-u) \end{pmatrix}$$

we have $-ut = -ut'\gamma(-u) \implies t = t'\gamma(-u)$. By the isotropy of space (time should transform the same way regardless of what direction u is), we should also have that $t = t'\gamma(u)$. This implies that $\boxed{\gamma(u) = \gamma(-u)}$. The same argument can be repeated with the equation between elements of the first components, and we get that $\theta(-u) = \theta(u)$.

2.4.2 Finding $\theta(u)$ and $\tau(u)$ in terms of $\gamma(u)$

- Now let's see what we can get if we expand the expression of the second condition:

$$L_{-u}L_u \stackrel{!}{=} I$$
$$\begin{pmatrix} \theta(-u)\theta(u) + \tau(-u)(u\gamma(u)) & \theta(-u)\tau(u) + \tau(-u)\gamma(u) \\ (-u\gamma(-u))\theta(u) + \gamma(-u)(u\gamma(u)) & (-u\gamma(-u))\tau(u) + \gamma(-u)\gamma(u) \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

looks very scary but its just ordinary matrix multiplication. These are four equalities. Let's look at the equality in the left-bottom corner of the matrix:

$$(-u\gamma(-u))\theta(u) + \gamma(-u)(u\gamma(u)) \stackrel{!}{=} 0$$

Recall that $\gamma(u) = \gamma(-u)$. We can then solve for $\theta(u)$ in terms of $\gamma(u)$:

$$-u\gamma(u)\theta(u) + u\gamma^2(u) \stackrel{!}{=} 0 \implies \boxed{\theta(u) = \gamma(u)}$$

- Now let's look at the bottom-right corner and try to solve for $\tau(u)$. We find that:

$$(-u\gamma(-u))\tau(u) + \gamma(-u)\gamma(u) \stackrel{!}{=} 1 \implies -u\gamma(u)\tau(u) + \gamma^2(u) = 1$$
$$\implies \boxed{\tau(u) = \frac{\gamma^2(u) - 1}{u\gamma(u)}}$$

- We can now express the entire matrix in terms of $\gamma(u)$!

$$L_u = \begin{pmatrix} \gamma(u) & \frac{\gamma^2(u)-1}{u\gamma(u)} \\ u\gamma(u) & \gamma(u) \end{pmatrix}$$

now all is left to determine L_u is to determine the function $\gamma(u)$.

2.4.3 Finding the function $\gamma(u)$

- Taking the third condition, which is essentially invoking the second postulate of special relativity (that an object moving at the speed of light should do so in all reference frames):

$$\begin{aligned} L_u \begin{pmatrix} t \\ ct \end{pmatrix} &\stackrel{!}{=} \begin{pmatrix} t' \\ ct' \end{pmatrix} \\ \begin{pmatrix} \gamma(u) & \frac{\gamma^2(u)-1}{u\gamma(u)} \\ u\gamma(u) & \gamma(u) \end{pmatrix} \begin{pmatrix} t \\ ct \end{pmatrix} &\stackrel{!}{=} \begin{pmatrix} t' \\ ct' \end{pmatrix} \\ \begin{pmatrix} t\gamma(u) + ct\frac{\gamma^2(u)-1}{u\gamma(u)} \\ tu\gamma(u) + ct\gamma(u) \end{pmatrix} &\stackrel{!}{=} \begin{pmatrix} t' \\ ct' \end{pmatrix} \end{aligned}$$

we have two equalities and two unknowns $\gamma(u)$ and t'. Either plug them into Worlfram alpha or solve them. Plugging the expression for t' (the first equality) in the second equality:

$$\cancel{t}u\gamma(u) + \cancel{ct\gamma(u)} = \cancel{c(t\gamma(u))} + c(c\cancel{t}\frac{\gamma^2(u)-1}{u\gamma(u)})$$

$$\implies \boxed{\gamma(u) = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}}$$

we have thus fully determined the new transformation matrix L_u.

- Let's keep using the variable $\gamma(u)$, but let's try to simplify the term in the upper right corner of L_u: $\frac{\gamma^2(u)-1}{u\gamma(u)} = (\frac{1}{u} - \frac{1}{u\gamma^2(u)})\gamma(u) = (\frac{1}{u} - \frac{1-\frac{u^2}{c^2}}{u})\gamma(u) = \gamma(u)(\cancel{\frac{1}{u}} - \cancel{\frac{1}{u}} + \frac{u}{c^2}) = \frac{u}{c^2}\gamma(u)$
- We finally get:

$$\boxed{L_u = \begin{pmatrix} \gamma(u) & \frac{u}{c^2}\gamma(u) \\ u\gamma(u) & \gamma(u) \end{pmatrix}}$$

- The function $\gamma(u)$ is called the *Lorentz factor*, and the transformation L_u is called the *Lorentz transformation*.
- **Excercise:** Check that the last two conditions do indeed hold, and find the new formula for addition of velocities.
- **Excercise:** The fact that c is the same in all reference frames implies that an object that's not already moving at c can not be moving at c in any reference frame. Check that this is indeed the case by trying to compute L_c.

2.5 Sorting the mess: Rewriting things neatly to match other literature

- **First, we need to sort out one mess**. We have been talking about L_u as the transformation that takes you from your frame to another frame in which according to it you're moving at velocity u. This was easier to calculate, but not so easy to phrase. In other literature, you'll see the parameter of the transform u being the velocity with which that frame is moving relative to your frame. In other words, they define $\Lambda_u = L_{-u}$. We'll switch to that convention from now on.
- We explicitly write Λ_u:

$$L_{-u} = \boxed{\Lambda_u = \begin{pmatrix} \gamma(u) & -\frac{u}{c^2}\gamma(u) \\ -u\gamma(u) & \gamma(u) \end{pmatrix}}$$

- Let's see how arbitrary events (t, x) transform under the lorentz transformation Λ_u that takes us to frame S' that's moving at u:

$$\Lambda_u \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma(u) & -\frac{u}{c^2}\gamma(u) \\ -u\gamma(u) & \gamma(u) \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma(u)(t - \frac{u}{c^2}x) \\ \gamma(u)(x - ut) \end{pmatrix} \overset{!}{=} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

- Suppose you have in your frame two events both at the same time t but one at position x and the other at position $x + \Delta x$. One can think the distance between them is:

$$\begin{pmatrix} t \\ x + \Delta x \end{pmatrix} - \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ \Delta x \end{pmatrix}$$

- What would be the distance in the reference frame S'? We need to transform both events and then compute the difference again. By linearity of matrices, it's the same as taking the difference and then computing:

$$\Lambda_u \begin{pmatrix} t \\ x + \Delta x \end{pmatrix} - \Lambda_u \begin{pmatrix} t \\ x \end{pmatrix} = \Lambda_u (\begin{pmatrix} t \\ x + \Delta x \end{pmatrix} - \begin{pmatrix} t \\ x \end{pmatrix}) = \Lambda_u \begin{pmatrix} 0 \\ \Delta x \end{pmatrix}$$
$$\implies (\Delta x)' = \begin{pmatrix} -\frac{u \Delta x}{c^2} \\ (\Delta x) \gamma(u) \end{pmatrix}$$

we have a 'spacital distance' $(\Delta x)\gamma$, but it doesn't help because the events are no longer simultaneous (!). We have a distance in space as well a distance in time in the new frame! Taking the spacial distance wouldn't make sense because it would be the difference between two events at two different times.

- What does that tell us? Two events that are simultaneous in one inerital frame may not be in the other. So "what happened first, a chicken or an egg?" might have different answers in different reference frames.

- In general, there is a trade-off between length contraction and simultaneouity. In most special relativity "paradoxes", the trick is to remember that.

- Now that we know time differences can change, how does the time interval between two events at the same position $(\Delta t + t, x)$ and (t, x) look like in the a different frame S'? In the current frame, the difference is just $(\Delta t, 0)$. Let's Lorentz transform it:

$$\Lambda_u \begin{pmatrix} \Delta t \\ 0 \end{pmatrix} = \begin{pmatrix} \Delta t \gamma(u) \\ -u \Delta t \gamma(u) \end{pmatrix} \overset{!}{=} \begin{pmatrix} \Delta t' \\ \Delta x' \end{pmatrix}$$

we see that here the spatial distance changes, but that does not matter for time measurement. Therefore, $\Delta t' = \Delta t \gamma(u)$. We have that $\gamma(u) \geq 1$, which means that time dialates when moving; or as they say, *moving clocks tick faster*. This effect is known as *time dialation.*

- A clock at rest is one constructed from events which x component has no velocity term ut for any velocity u involved. If a clock at rest measures the (proper) time interval $\Delta\tau$, then in another moving frame, it would measure $\Delta t = \tau\gamma(u)$ as we have seen above due to time dialation. This implies $\boxed{\Delta\tau = \frac{\Delta t}{\gamma(u)}}$. This is an invariant quantity that's the same in all frames.

- But if time is changing in every frame, what would be the right time to measure? It's simply the time where the clock is at rest. By definition, it's the same in all reference frames. We call it the *proper time* and we define it by $\Delta\tau$ (or sometimes τ).

- With a similar idea, we can define a proper length L in frame that is at rest. One can show in a similar way to proper time that it's $L_0 = \gamma(u)L$ once they have found length contraction.

- **Excercise:** Show that *length contraction*; show that the $\Delta x' = \frac{1}{\gamma(u)}\Delta x$ where $\Delta x'$. Avoid simultaneity issues we encounted before by ensuring $\Delta t' = 0$ in the new frame S'. Hint: Start with Δt arbitrary and then insist that $\Delta t' = 0$. After that, solve for Δt and substitute in the equation for $\Delta x'$.

2.6 Lorentz transformations in three dimensions

- So far, we have only worked with x position and time, but the world has three spatial dimensions.

- Claim: if motion is strictly on the x axis (no velocity term for y and z), transformations for y and z are trivial: $y = y'$, $z = z'$.

- Proof: Since Lorentz transformations are linear, they must map $y = 0$ to $y' = 0$, and $z = 0$ to $z' = 0$. If $y, z \neq 0$, we just translate everything so that they're at zero. Changing the origin of the coordinate system should not change its physics (in general: changing the coordinate system shouldn't affect the physics. That's the principle of *general covariance*).

- So we have:

$$\Lambda_u \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma(u) & -\frac{u}{c^2}\gamma(u) & 0 & 0 \\ -u\gamma(u) & \gamma(u) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma(u)(t - \frac{u}{c^2}x) \\ \gamma(u)(x - ut) \\ y \\ z \end{pmatrix} \overset{!}{=} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix}$$

- What if u is not strictly in the x direction? We rotate everything so that u is in the x-axis direction with the rotation matrix R^{-1}, do our physics (lorentz transform), and then rotate back with R. This is equivalent to just hitting L_u by "conjugation" of R (means, hit on the right with inverse, and on the left with itself):

$$\Lambda_{R,u} = R\Lambda_u R^{-1}$$

and then use the new $\Lambda_{R,u}$.

2.7 Lorentz covariance

- In physics, we deal with quantities. But we need good ones that are well defined and consistent. It's a bit vague at the moment, but soon it should be made clear.

- Scalars quantities like the spatial distance between two events Δx are bad. It takes different values in different inertial frames. A good quantity is one that's *Lorentz invariant* a quantity that is the same in all inerital frames. For example, the proper time $\Delta\tau$ is a good quantity and is Lorentz invariant. It's the same in all reference frames. Other examples of Lorentz invariant quantities are the rest mass m_0, the speed of light c, the proper length and so forth.

- Any meaningful vector quantities which involve the physical dimensions Length or Time will have to transform under a lorentz transformation. We base that of the fact that an event that describes a position-time in spacetime transforms by application of the Lorentz transform operator L_u. Velocity for instance is then the derivative of this position, and momentum would be a mass times this position ..etc.

- However, this one disturbing thing with having an event as a quantity of position. A quantity should have one physical dimension (Length, Time ..etc). However, the first component in the event vector has dimension Time, and the rest have dimension Distance/Spactial-Position.

- To make everything consistent, we must make the first components of an event also have a dimension of Spacial-Position. We know the way to make a distance from a time quantity is to multiply it by a velocity quantity.
- In order for the event to still transform similarly, the velocity we multiply by time better be an invariant, and we know that got to be c. This way, it will simply act as a scaling factor.
- We define the *4-Position*:

$$X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

our base position quantity in special relativity. To account for the scaling in time, the Lorentz transformation changes form:

$$\Lambda = \begin{pmatrix} \gamma(u) & -\frac{u}{c}\gamma(u) & 0 & 0 \\ -\frac{u}{c}\gamma(u) & \gamma(u) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

one can easily check that applying this on the the 4-Position vector X gives the same equations when an event without a scaling factor c is hit by the old Lorentz transform.

- One can try to define a velocity out of this. From the formal definition of a derivative, $\frac{dX}{dt} = \lim_{\Delta t \to 0} \frac{\Delta X}{\Delta t}$. The difference between two 4-vectors will transform by multiplication by Λ under a change of inertial frames:

$$\begin{aligned} & \Delta X = X_1 - X_2 \\ \overset{\text{go to frame } S'}{\rightarrow} \; & \Delta X' = X_1' - X_2' = \Lambda X_1 - \Lambda X_2 = \Lambda(X_1 - X_2) = \Lambda \Delta X \\ \overset{\text{derivative limit}}{\rightarrow} \; & dX' = \Lambda dX \end{aligned}$$

However, Δt is a scalar that's not a lorentz scalar, so will change. Instead, we use the proper time $\Delta\tau$:

$$V = \frac{dX}{d\tau}$$

- Since $d\tau$ is invariant, V transforms like dX which transforms like X with multiplication by Λ.

- We can express $d\tau$ as follows:

$$\Delta\tau = \frac{1}{\gamma}\Delta t \implies d\tau = \frac{1}{\gamma}dt$$

$$\implies \boxed{V = \gamma\frac{dX}{dt} = \gamma\begin{pmatrix} c \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}}$$

- Similarly, we can define other quantities like 4-momentum as $P = m_0V = m_0\gamma\frac{dX}{dt}$ where m_0 is the rest mass, a Lorentz invariant. And then the 4-force $\frac{dP}{d\tau} = \gamma\frac{dP}{dt}$, which is also a four vector. And the list goes on. These are all quantities that respect Lorentz transformation.

- In general, if all quantities in a law respect lorentz transformations, then the law is lorentz covariant and is compatable with special relativity.

- Maxwell equations, in particular, are lorentz invariant; SR was based to make Maxwell's equations hold afterall. We know how the velocity of light transforms for example; it's a lorentz invariant. One can also check that other quantities form nice four vectors and everything. More will be taught in the second year course "Electrodynamics".

- Newton's law fo gravitation however, does not respect lorentz transofrmations. In particular, it has explicit dependence on the distance between two bodies r, which is not a lorentz invariant nor a lorentz covariant.