The General Relativity Marathon

Warming up

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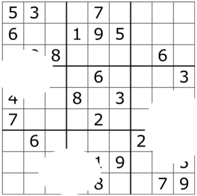
Physics: As a sudoku puzzle

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

- Main strategy: **Restriction** Another strategy: **Guessing** Or maybe the other way around. Or a mixture of both. Both have been very fruitful in physics, and different physicists have different styles
- Instances when guessing was very successful: Schrodinger equation of quantum mechanics. It solved a biig chunk of the board!
- Instances when guessing worked partially: Dirac Quantisation
- Instances when restriction worked wonders: Quantum Electrodynamics

Nature as the grid



- We don't think we see all the grid yet. Seeing more of the grid $\leftarrow$ doing new experiments
- Main condition for the puzzle: **Agree with all experiments**
- Example of a simple restriction technique: Dimensional Analysis
- Other example: Enforcing sensible invariance conditions (we'll see later today)

Degrees of Freedom

# of degrees of freedom: # of real numbers you need to fully specify a system
Or more simply: Number of choices you can make.

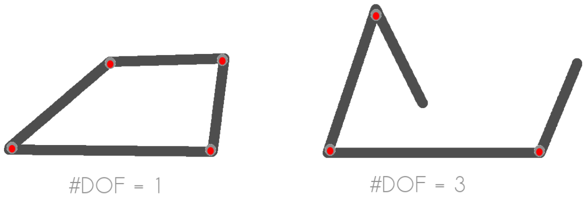


Figure: Assume the horizontal base is locked in position and orientation

Notice how adding an extra joint reduced the degrees of freedom

Degrees of freedom

- Some theories show extra degrees of freedom that do not change the outcome of any experiment you can do; such theories are called *Gauge theories*. The freedom of such choice is called *Gauge freedom*, and the process of changing such parameter is called a *Gauge transformation*
- Example: The electric potential at a point $\phi(x)$ is a *gauge potential*. The only thing that matters in experiment is the potential difference, and a transformation $\phi(x) \to \phi(x) + C$ doesn't affect the difference
- In this sense, a quantity that changes under a gauge transformation cannot be *physical* (the issue is more complicated. For more info, read on the Aharonov-Bohm effect)
- The condition that a physical law must not change under a gauge transformation is called *gauge invariance*

More on degrees of freedom, Invariance

- Notions of invariance has very deep consequences in physics
- A continuous symmetry is a continuous family of transformations that keep equations of motion invariant
- Analytical Mechanics spoiler: *Noether's theorem* says that continuous symmetries $\rightarrow$ conserved quantities. e.g.: Time symmetry $\rightarrow$ Energy is conserved!
- In quantum field theory, gauge freedoms are eliminated by introducing 'gauge bosons' that eats the degrees of freedom (analogue to: introducing a joint in the example from before). Gauge potential in EM $\rightarrow$ photons are introduced!
- If your theory is a gauge theory with lots of degrees of freedom, it would predict lots of particles to cancel them out. If such particle are not observed in nature, your theory is bad
- One of the major challenges of making a quantum theory of gravity is that naively quantising GR introduces extra degrees of freedom that shouldn't be there

Gedanken experiment: Let the sun begone!

Newton's law of gravitation says that the magnitude of the force acting on the earth from the the sun is:

$$F_G = \frac{GM_{sun}m_{earth}}{(x_{earth} - x_{sun})^2}$$

If the sun suddenly disappears, it tells us that since $M_{sun} = 0$ the earth immediately stops feeling any force and leaves orbit. But this is in contradition with special relativity that tells us that it would stop feeling the force only after 8 minutes have passed. We'll see later that whether a theory agrees with special relativity is a question of whether its equations are 'Lorentz Invariant'; i.e. invariant under a change of 'inertial frame of reference' consisting of rotations and 'boosts' (as we shall see later in this seminar). So Newton's theory of gravity needs to be modified to be Lorentz invariant.

Locality

An often desired property for lorentz invariant theories is *locality*. This means that any observable at x must only be a function of x or its immediate neighbohrhood $(x + \delta x)$. Newton's theory of gravity shown above lacks locality because the force is a function of two distant points x_{earth} and x_{sun}. Intuitively, locality means that there's no 'interaction at a distance' and that interactions and changes only propagate. However there do exist theories that are non-local but are still lorentz invariant. It's just very tricky to have that.

Classical field theories

A field theory has two main components: A field law that gives you a field and a force law to give you $\vec{F}$. The dynamical variable in field laws are usually the *potentials*; i.e. the potential is the solution, and from it one can get everything else.

In Newton's formalism of classical mechanics, the main equation of motion is

$$\text{Newton's law:} \quad \vec{F} = m\ddot{\vec{x}}$$

where m is the *inertial mass* of a test particle, and F is some arbitrary force as a result of some interaction. Solving the differential equation gives one the trajectory of the particle as a function $x(t)$.

Example of a local, lorentz invariant theory: Maxwell's

Maxwell's theory of electrodynamics, assuming no magnetic interaction here, gives:

$$\vec{E}_E = -\nabla\phi_E \qquad \nabla\cdot\vec{E}_E = \frac{1}{\varepsilon_0}\rho_E$$

where ϕ_E is the electric potential, $\vec{E}_E$ is the electric field and ρ_E is the electric charge density.
Then the force $\vec{F}_E$ on a test particle with electric 'mass' (electric charge) q is $\vec{F}_E = q\vec{E}_E$.
Maxwell's theory says that if a charge suddenly vanishes, the effect would propagate as an electromagnetic wave with the speed of light c, but that requires the magnetic counterpart.
Assuming a point like source with charge Q, we can produce columb's law:

$$\phi_E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \implies \vec{F}_E = q\nabla(\frac{1}{4\pi\epsilon_0}\frac{Q}{r}) = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}$$

Newtonian gravity in terms of fields

Let's try to formulate Newton's gravity in a similar way so that it also produces newton's equation of gravitation for a point source:

$$\vec{E}_G = -\nabla\phi_G \qquad \nabla \cdot \vec{E}_G = -4\pi G\rho_E$$

where ϕ_G is the gravitational potential, $\vec{E}_G$ is the gravitational field and ρ_G is the 'gravitational charge' (gravitational mass) density.
Then the force $\vec{F}_G$ on a test particle with gravitational 'mass' m_G is $\vec{F}_G = m_G\vec{E}_G$.
This would produce the equations we want, but it's not lorentz invariant until we introduce some notion of 'Gravitomagnetic field'.

A geometric theory of gravity

We could easily introduce a gravitomagnetic field with equations analogue to Maxwell's and make it lorentz invariant. But Einstein noticed something deeper that would later result in a much better theory that would explain some of the anomalies a Newton-like theory wouldn't be able to explain (the problem of Mercury's orbit precession).

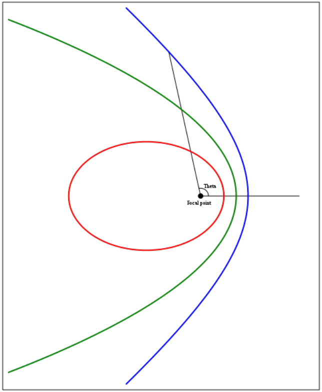
Using $F = m\ddot{x}$, the path a particle takes in the presence of an electric field is the result of setting $F = F_E$, and similarly for a gravitational field by setting $F = F_G$:

$$\ddot{x} = \frac{qE_E}{m} \qquad \ddot{x} = \frac{m_G E_G}{m}$$

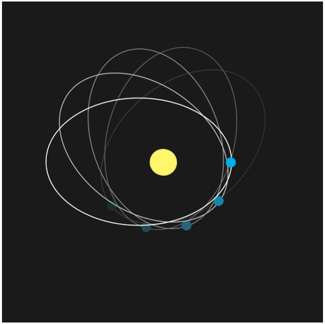
From Newton's gravity, it's always dealt with as if the gravitational mass equals the inertial mass: $m_G = m \implies \ddot{x} = E_G = -g$. This was a hint that Newton's second law is not needed at all: gravity is does not require a 'force'. The field acceleration acts of objects directly. Gravity is a *geometrical* in nature.

Success stories of GR: Precession of Mercury

Newton's theory of gravity predicts 'keplarian orbits' (orbits never cross; elliptical). Reality: Mercury's orbit isn't keplarian; it precesses. We shall see that GR deals with that!



(a) Keplarian Orbits



(b) Orbit of Mercury; not a perfect ellipse so it precesses

Success stories of GR: Gravitational lensing

GR predicts that light gets bent by gravity. It's been observed in eclipses, but it's even more vividly manifest in the exquisite 'Einstein Rings'. We'll see that later in more detail

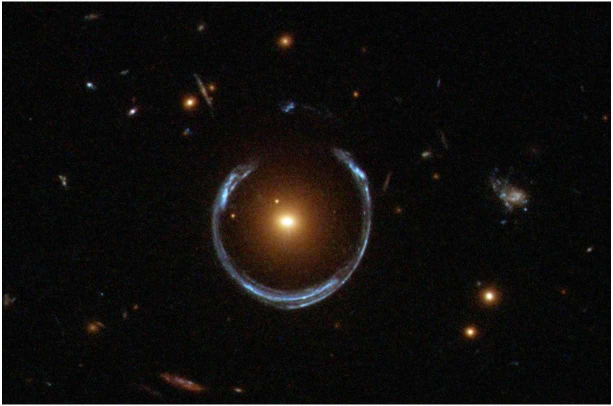


Figure: Picture from Hubble's telescope. Red galaxy bends the light of a farther blue galaxy behind it; much like a lens!

Success stories of GR: Gravitational waves

GR predicts the disturbance from a change in the gravitational field can be decoupled into a volume changing component (Ricci curvature tensor) and a tidal force component that only distorts and propagates (Weyl tensor), much like a wave! This was detected recently in 2015 by the LIGO detector

General Relativity

General relativity is greatly constrained by three restrictions:
1- Dimensional Analysis
2- General Covariance: Invariance under coordinate transformations)
3- Lorentz Invariance: Invariance under change of inertial frames of reference
This gives us very little choice in the so called ‘action formulation’ of general relativity. The Einstein Hilbert action is given by:

$$S = \frac{1}{2\kappa} \int R \sqrt{-g} \, \mathrm{d}^4 x$$

And it produces: $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$

Additionally we have: $\dfrac{\partial^2 x^\lambda}{\partial \tau^2} = -\Gamma^\lambda_{\mu\nu} \dfrac{\partial x^\mu}{\partial \tau} \dfrac{\partial x^\nu}{\partial \tau}$

Objects in the Einstein field equation(s)

- $g_{\mu\nu}$ is the 'metric'; an object that defines the notion of distances at each point on spacetime. It's our dynamical variable.
- R the ricci scalar; quantifies magnitude of curvature. In a sense: the (double trace of) the second derivative of the metric.
- $T_{\mu\nu}$ is the 'stress energy tensor'. It's the conserved quantity under space-time translation, and it accounts for matter sources on the spacetime.
- $G_{\mu\nu}$: the 'Einstein tensor' a function of curvature tensors ('second derivatives of the metric') that's been selected by restriction of 'energy-stress-momentum conservation' and other conditions.

In a sense, EFE is an analogue of the 'poisson equation' with $g_{\mu\nu}$ as the dynamical variable instead of a potential ϕ, and $T_{\mu\nu}$ playing the role of the mass/charge density. We'll look into all these objects in detail later.

Some geometry: distance

- Distance between two points: length of the *shortest curve that connects two points*.
- Shortest curve that connects two points is called the *geodesic*.

Distance between two points on a plane Δs:

$$\Delta s^2 = \Delta x^2 + \Delta y^2$$

What if surface is not flat? e.g: distance between points on surface of a balloon
The distance is easier defined locally, only in the immediate neighbohrhood.
We'll see more later on.
Next session: Special Relativity!