Notes: Hens in the Dimensions (Talk on Dimensional Analysis)

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1 Quantities, Dimensions

1.1 Base Quantities

- 10 kg, 30 s; these are examples of quantities.
- The unit 'kg' tells you that the first quantity has dimension of mass. The unit 's' tells you that the second quantity has dimension of time.
- If $a = 10$ kg, $b = 1$ m and $b = 30$ s, then we use square brackets to denote the dimension of the quantity:

$$
\begin{aligned}
[a] &= M \\
[b] &= L \\
[c] &= T
\end{aligned}
$$

- Distinguish a *unit* from a *dimension*. 3 seconds and 1 minute are two quantities of different units (seconds and minutes), but same dimension (Time or T)
- So formally: a quantity is a *number equipped with a dimension* (not a unit).

1.2 Derived Quantities

- If you're walking at $v = 0.5$ m/s, then v has a dimension of velocity.
- But velocity can be expressed in terms of L and T. So $[v] = LT^{-1}$.

- In the above sense, L, M and T are base dimensions since you can't form one from another. They're *independent*. Whereas velocity has a derived dimension.
- Alternatively, the base dimensions could have been defined to be L, M and V, then T could have been the derived dimension LV^{-1}.

1.3 Principle of Dimensional Homogeniety

- "My mass in kilograms is equal to my height in metres!" is a funny statement that could as well be true.
- It corresponds to the equation $m = h$, but if we decide to use kilometres instead of metres, it would no longer hold.
- But laws of physics should invariant under a change of units. This can only be achieved if both sides of a comparison have the same dimension.
- This is the *principle of dimensional homogeniety*.
- Similarly, saying that the mass of a wire plus its length gives its current is not a physical relation because it can change with a change of units; hence, one can only add quantities of the same dimension.

2 Operations on Dimensions and Bridgmann's Theorem

2.1 Dimensions as a Vector Space

- Some examples of other derived quantities and their dimensions:

$$\begin{aligned} p &= mv && [p] = \text{Momentum} = M^1L^1T^{-1} \\ a &= vt && [a] = \text{Acceleration} = L^1T^{-2} = M^0L^1T^{-2} \\ f &= ma && [f] = \text{Force} = M^1L^1T^{-2} \\ E &= mc^2 && [E] = \text{Energy} = M^1L^2T^{-2} \end{aligned}$$

- So we seen in general that quantities related to mechanics generally have a dimension $M^{\alpha_1}L^{\alpha_2}T^{\alpha_3}$.

- If we had dealing with electromagnetism, we would just need to introduce one more base dimension I for current.
- We know that when two quantities are added, they must have the same dimension, and the sum will still have the same dimension; My height plus my shoes height will still be of dimension L.
- Let's observe what happens to the dimensions when we multiply and divide two quantities together (l is some length, d is some distance and v is some velocity):

$$\begin{aligned} l \cdot l &= l^2 & \quad [l \cdot l] = M^0 L^{1+1} T^0 = L^2 \\ 1/d &= d^{-1} & \quad M^0 L^{-(1)} T^0 = L^{-1} \\ v/d &= v \cdot d^{-1} & \quad [vd^{-1}] = T = M^0 L^{1+-1} T^{-1} = T^{-1} \end{aligned}$$

In general, given two quantities Q_1 of dimension $M^{\alpha_1} L^{\alpha_2} T^{\alpha_3}$ and Q_2 of dimension $M^{\beta_1} L^{\beta_2} T^{\beta_3}$:

$$\begin{aligned} [Q_1 \cdot Q_2] &= M^{\alpha_1+\beta_1} L^{\alpha_2+\beta_2} T^{\alpha_3+\beta_3} \\ [Q_1^{-1}] &= M^{-\alpha_1} L^{-\alpha_2} T^{-\alpha_3} \\ [Q_1^n] &= M^{n\alpha_1} L^{n\alpha_2} T^{n\alpha_3} \end{aligned}$$

- If M, L and T letters were invisible, and you write only the exponents in a funny way:

$$[Q_1] = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

It looks like a 3D vector. Multiplication of two quantities corresponds to addition of their two vectors:

$$[Q_1 \cdot Q_2] = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \alpha_3 + \beta_3 \end{pmatrix} = M^{\alpha_1+\beta_1} L^{\alpha_2+\beta_2} T^{\alpha_3+\beta_3}$$

You'd also see that division corresponds to subtraction of the dimension vectors, and exponentiation corresponds to multiplying the dimension vector by a scalar. The zero vector is the dimension of a dimensionless quantity like the ratio between two lengths.

- We say the space of dimensions forms a *vector space* under multiplications of quantities and their exponentiation, and the base dimensions are the "basis" vectors.
- The study of vector spaces is one of the main topics of Linear Algebra, so we can use all tools and theorems of vector spaces developed in mathematics to work with dimensions.

2.2 Exponentiation, trigonometric function, logarithms..etc of Quantities

- One of the ways to define e^x is as an infinite series:

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- We have seen before that taking powers of a quantity multiplies it dimension vector by the exponent. The only vector that doesn't change under multiplication of scalars is the zero vector, which corresponds to the zero dimension.
- Addition is only allowed for quantities of the same dimension. So we can conclude that if we have a physical law that involves exponentiation of a quantity e^Q, then Q must be dimensionless.
- In fact, given an arbitrary (analytic) function $f(x)$, it can be expressed as a taylor series:

$$f(x) = f((x-h)+h) = f(x-h) + h\frac{f'(x-h)}{1!} + h^2\frac{f''(x-h)}{2!} + \cdots$$

for most function, this produces a sum of different powers of h (which should have the same dimension of x). This includes $\sin(x)$, $\cos(x)$, ln and polynomials with more than one term (like $x^2 + x$).

2.3 Bridgmann theorem

- One can easily prove that the only form of functions that can take a dimensionful parameter x are of the form:

$$f(X) = CX^n$$

where X is a dimensionful (or a dimensionless) quantity, and C, n are (dimensionless) numbers.

- Note that X may also be composed of other quantities; e.g: $v(a,t) = 1 \cdot (u + at)^1$. In this case, $X = u + at$, $n = 1$ and $C = 1$.
- This statement is called *Bridgmann's Theorem*, and it tells us that the form of physical laws must be as such.

3 Use cases and examples of Dimensional analysis

3.1 Sanity checking

- Dimensional analysis is useful for sanity checking your physical laws. A physical law should satisfy the principle of dimensional homogeniety and addition/subtraction should only be between quantities of the same dimension.

3.2 Determining the dimension of an unknown

- Example: Given the following physical law, what is the dimension of the quantity Q?

$$v = \frac{\frac{Q}{m} + d}{t} + gt$$

where m is a mass, d is a distance, t is a time, and g is an acceleration. d has a dimension of distance, so the quantity $\frac{Q}{m}$ that you add to it must also have a dimension of distance. Division of two quantities is subtraction of their dimensions vectors. Writing the dimension of mass out, it's $[m] = M^1L^0T^0$ or $\begin{pmatrix}1 & 0 & 0\end{pmatrix}^T$, and $[d] = \begin{pmatrix}0 & 1 & 0\end{pmatrix}^T$. Let's denote the unknown $[Q] = \begin{pmatrix}x_1 & x_2 & x_3\end{pmatrix}^T$. So we need that:

$$[d] = [Q/m] \implies \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}$$

so the answer is $[Q] = \begin{pmatrix}1 & 1 & 0\end{pmatrix}^T$ or $M^1L^1T^0$.

3.3 Determining the form of new physical laws

- This works if you assume you know all the physical quantities that characterize a problem.

3.3.1 The period of a pendulum

- Four quantities in play: gravity g, mass of pendulum m, length of the pendulum l and the initial angle θ_0. $[l] = L$, $[m] = M$, $[g] = ML^{-2}$ and θ_0 is dimensionless; $[\theta_0] = M^0 L^0 T^0 = 1$.

- We're interested in finding the period T of the pendulum which has the dimension of time.

- We want $T(\theta_0, g, m, l)$. Bridgmann's theorem tells us it must be of the form:

$$T(\theta_0, g, m, l) = C(\theta_0) X(\theta_0, g, m, l)^n$$

where X^n is a combination $g^\mu m^\nu l^\xi$ that has dimension of time.

- The dimensionless quantity C can only depend on dimensionless quantities, and the only one we can form here is θ_0

- We need that:

$$[g^\mu m^\nu l^\xi] = \mu \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + \nu \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \xi \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \overset{!}{=} [T] = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

which solving gives the *unique* solution $\mu = -\frac{1}{2}$, $\nu = 0$ and $\xi = \frac{1}{2}$.

- This tells us that $X^n = \sqrt{\frac{l}{g}}$, and that $T = C(\theta_0)\sqrt{l}g$

- The solution is correct upto a dimensionless quantity that can only depend on the initial angle! (we know that to be 2π) . It's still useful, for instance, in computing the ratio of the period of two different pendulums with the same initial angle θ_0.

3.4 The atomic bomb

- G.I. Taylor had a photo with a scales on it.
- Quantities were identified to be the air density ρ, air pressure p, the shock front radius R and the time from explosion T.
- Taylor assumed in case of an explosion as powerful, the pressure of air outside would be negligible.
- He was interested in the energy E released. Solving the same problem as above, $[E] = ML^2T^{-2}$.
- Similarly to above example, there's only one form this function can take, which gives:

$$E = C\frac{\rho R^5}{t^2}$$

where C is a constant.

- This allowed Taylor to make his estimate of 22 kilotons of TNT, which was so close to the true value 20 kilotons of TNT.

3.5 Integral of a Gaussian

- Sometimes one can "inject dimensions" into problems that are not necessarily physics related to be able to use the tools of dimensional analysis.
- Example, what's the integral $\int_{-b}^{b} Ae^{-ax^2}dx = A\int_{x_2}^{x_1} e^{-ax^2}dx$ for big b
- The integral is an "area under a curve". Perhaps let's assume this area has dimension L^2
- dx are small intervals in the x-axis, so $[dx] = L$. Ae^{-ax^2} is the y-axis value, so it should also be given a dimension of length.
- But we know from before e^x is dimensionless, so $A but that's no problem if we assume A that was extracted from the integral has dimension of length.
- The argument of e^x must always be dimensionless. $[x] = L$, therefore $[a]$ must be L^{-2} .

- The integral decays rapidly to zero away from the center, so the integral's dependence on b can be neglected.
- On the right hand side, we'll have A, given a dimension L, since it comes out of the integral. Thus we only have a to use in order to form a quantity of dimension L^2 on the other side.

$$[\int_{-b}^{b} Ae^{-ax^2} dx] = L^2 = [A][a^\gamma]$$
$$\iff (2) = (1) + \gamma(-2) \implies \gamma = -\frac{1}{2}$$

which tells us that the integral must be of the form $CA\frac{1}{\sqrt{a}}$, where C is dimensionless constant.

- There are some very nice examples on the use of dimensional analysis in pure mathematics in a blog post by Tao referenced below.

4 Dimensionless Quantities

- Dimensionless quantities are important because they do not change under a change of units; they don't scale.
- Dimensionless quantities are the only ones where it makes sense to speak about its scale.
- i.e. if your equation only works when $Q << 1$, then Q better be a dimensionless quantity!
- MUST ALWAYS be dimensionless
- $\sin(\theta) \approx \theta$ when $\theta << 1$ is OK since θ is dimensionless.
- If it's not, then there should at least be another quantity that's compared to this. For example, a specific formula might work when the "height of the room h is small compared to its width w". In which case, you're really saying that $\frac{h}{w} << 1$, and $\frac{h}{w}$ is dimensionless.

5 Buckingham π Theorem

- Claim: one can express any law in a form where both right hand side and left hand side are dimensionless.
- Proof: Take any law. e.g: $F = ma$. Then $\frac{F}{ma} = 1$ obviously has both its right hand side and left hand side dimensionless.
- Buckingham π Theorem statement: The number of dimensionless quantities you can form p equals the number of quantities in a problem n minus the number of independent base dimensions used k.
- Very powerful statement.
- As a collary, once you've determiend all these dimensionless quantities, you can rewrite any law in terms of them.
- Another corollary, if a problem has only one dimensionless quantity associated with it, this quantity must be a constant (since you need to equate it to something dimensionless, but if its alone, then you need to equate it to a dimensionless constant).
- Proof of the theorem, essentially rank-nullity theorem from linear algebra.

5.1 Derivation of Kepler's third law with Dimensional Analysis and Buckingham

- The variables that characterise a planet's orbit are the the mass m, its distance from the sun r its period t and the force F acting on it.
- The only dimensionless quantity we can construct is:

$$\frac{Ft^2}{mr}$$

Since any law can be expressed with both sides being a function of dimensionless variables, one side must depend on it, and one side must be a constant. So this must be a constant and must hold for any planet. We denote it by k below.

- Apply to the force law:

$$F = G\frac{Mm}{r^2} \implies \frac{t^2}{mr} \cdot F = \frac{t^2}{mr} \cdot G\frac{Mm}{r^2} = k$$

- This gives $\frac{t^2}{r^3} = \frac{k}{GM} = \text{const}$, or that $t^2 \propto r^3$. That's Kepler's third law!

6 References and Further Reading:

1. "Fluid Dynamics for Physicists" by T.E. Faber
2. "The Physical Basis of Dimensional Analysis" online pdf by Ain. A Sonin
3. "Dynamics and Relativity" lecture notes chapter three by David Tong
4. "Street Fighting Mathematics" ebook by MIT
5. Derivation of kepler's third law with Dimensional Analysis with dimensional analysis from physics stackexchange
6. "Amplification, arbitrage, and the tensor power trick," blog article by Terrence Tao