Notes: Evil Quaternions and a Theory of Electrothermomagnetism

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These notes were written for a talk for Jacobs student Physics Society on the paper *Physical Space as a Quaternion Structure, I: Maxwell Equations. A Brief Note.* by P. Jack. These notes in addition also include an introduction to quaternions. We shall stick to the Gaussian units throughout as the main paper did, wherein the electric field and the magnetic field have the same units.

1 A quick overview of Maxwell's equations with Heaviside-Gibbs vectors

- In the 19th century, James Clerk Maxwell successfully unified electricity with magnetism as is portrayed in the following set of equations (with $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\nabla\phi - \frac{1}{c}\partial_t \vec{A}$):

$$\nabla \cdot \vec{E} = 4\pi\rho \tag{1}$$

$$\nabla \cdot \vec{B} = 0 \tag{2}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\partial_t \vec{B} \tag{3}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\partial_t \vec{E} \tag{4}$$

  - Equation (1): Electric field is sourced by electric charges
  - Equation (2): Magnetic field is never sourced; there are no magnetic charges/monopoles
  - Equation (3): Farday's law; A change in the magnetic field induces a circulating (thus non-conservative, and hence capable of doing work) electric field (and vice versa).
  - Equation (4): Ampere's circuital law; Flow of current comes with a circulating magnetic field which plane is in the direction of the current's flow (and vice versa).

- The equations written above make use of Heaviside/Gibbs vectors, but initially, Maxwell wrote them down in terms of components and later in terms of quaternions.

2 An overview of thermoelectric effects

- Current passing through a material with conductivity σ always releases heat Q given by:

$$Q = J(\sigma^{-1}J)$$

  In this case, Q is called *Joule's irreversible heat.*

- The peltier/seebeck effect is best demonstrated in a diagram:

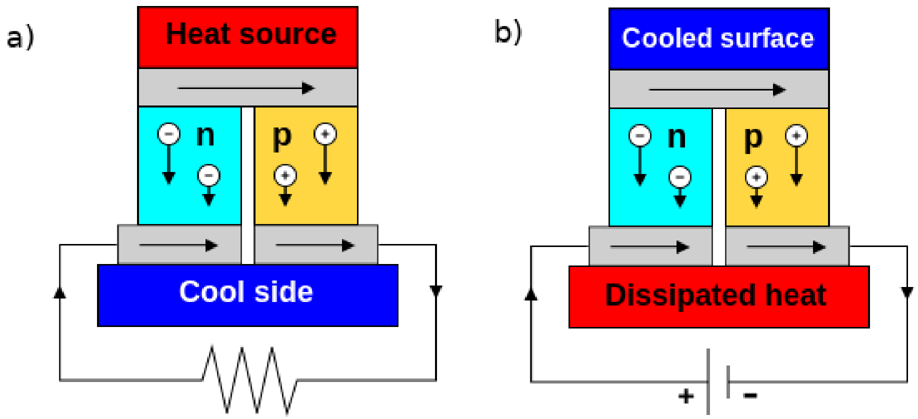

Figure 1: Configuration of a thermoelectric generator(a)/cooler(b) Ref: by CM Cullen (GFDL 1.2 and CC-by 2.5 licensed)

In figure a), current is generated due to negative charge carriers (electrons) in the n-doped part diffusing from the hot side to the cold side, and positive charge carriers (holes) in the p-doped part diffusing from the hot side to the cold side. This effect is called the "Seebeck effect" and the emf is quantified from the electric field given by:

$$E_{emf} = -S\Delta K$$

where S is the "seebeck coeffecient" and ΔK is the change in temperature.

In figure b), the effect is reversed by taking advantage of the symmetry of the situation where a current is passed instead to produce the tempearture gradient. The change in heat is given by:

$$\frac{dQ}{dt} = (\Pi_A - \Pi_B)I$$

where $\Pi = SK$ is the peltier coeffecient.

- Thomson effect: Discovered by William Thomson (Kelvin), and is a generalization of Peltier and Seebeck effects for non-uniform gradients of temperature:

$$\frac{dQ}{dt} = -\mathcal{K}J \cdot \nabla K \tag{5}$$

where $\mathcal{K}$ is the *thomson coeffecient*. The relations between the Seebeck coeffecient S, the Peltier coeffecient Π and the thomson coeffecient $\mathcal{K}$ are:

$$\mathcal{K} = K\frac{dS}{dK} = \frac{d\Pi}{dK} - S$$

3 The evil quaternion division algebra

- William Rowan Hamilton wished to extend complex numbers, a two dimensional number system, to a three dimensional number system where division is uniquely defined for all numbers except divison by zero [1], but he could not achieve that.

- As the story goes: In 16th of October 1843, while he was strolling with his wife, he arrived at the answer; It was not a three dimensioanl system however, but four.

- The complex numbers $z \in \mathbb{C}$ can be expressed as $z = x + y\mathbf{i}$, where x is called the *real part* and y is called the *imaginary part*. Alternatively, it can be written in polar coordinates as $z = [r, \theta] = re^{\theta\mathbf{i}}$, where $r \in \mathbb{R}^{\geq 0}$ is called the *modulus* (the norm of z) and $\theta \in [0, 2\pi)$ is called the *argument* of z. The main rule for multiplication of complex numbers is that $\mathbf{i}^2 = -1$.

- A quaternion $q \in \mathbb{H}$ can be expressed as $q = t + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where we have the multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

$$\mathbf{i} = \mathbf{jk} = -\mathbf{kj} \;, \mathbf{j} = \mathbf{ki} = -\mathbf{ik} \;, \mathbf{k} = \mathbf{ij} = -\mathbf{ji}$$

where t is called the *scalar part* and $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is called the *vector part* of q. A quaternion with a zero scalar part is a *pure quaternion*. Alternatively, one can be express q in "polar coordinates" with an extended exponential as:

$$re^{\theta(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = r[\cos\theta + (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\sin\theta]$$

where $r \in \mathbb{R}^{\geq 0}$ is called the *tensor part* and $e^{\theta(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} \in \mathbb{H}$ is a unit quaternion and is called the *versor part* (unit part) of q. It's clear that there is no unique "polar" representation for q, since we have 5 degrees of freedom in this representation $(r, \theta, u_x, u_y, u_z)$ contrary to the 4 degrees of freedom expected.

- Note also that multiplication of this exponential is not simply the addition of the arguments, since quaternion multiplication is not commutative. Instead we use the Baker-Campbell-Hausdorff formula:

$$\begin{aligned}\exp\{a\}\exp\{b\} = \exp\Big\{a + b + \frac{1}{2}[a,b] +& \\ &+ \frac{1}{12}\left([a,[a,b]] + [b,[b,a]]\right) \\ &- \frac{1}{24}[b,[a,[a,b]]] \\ &- \frac{1}{720}\left([b,[b,[b,[b,a]]]] + [a,[a,[a,[a,b]]]]\right) \\ &+ \frac{1}{360}\left([a,[b,[b,[b,a]]]] + [b,[a,[a,[a,b]]]]\right) \\ &+ \frac{1}{120}\left([b,[a,[b,[a,b]]]] + [a,[b,[a,[b,a]]]]\right) + \cdots\Big\}\end{aligned}$$

  so it's not as convenient.

[1] That is to say: it should form a division algebra

- Multiplication by a unit complex number $e^{\mathbf{i}\theta}$ rotates another anticlockwise by angle θ. In the case of quaternions, *conjugation* of a pure quaternion $p = p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k}$ by a versor $q = \cos\frac{\theta}{2} + (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\sin\frac{\theta}{2}$:

$$p' = qpq^{-1}$$

rotates the vector $\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$ on the axis defined by $\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ anticlockwise by an angle θ in a pure quaternion $p' = p'_x\mathbf{i} + p'_y\mathbf{j} + p'_z\mathbf{k}$ with $\begin{pmatrix} p'_x \\ p'_y \\ p'_z \end{pmatrix}$ being the new coordinates of the rotated vector.

- It's convenient to just write $q = t + \vec{v}$, where t is the scalar part and the components of the heaviside-gibbs vector $\vec{v}$ are the coeffecients of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ respectively. We can think of the scalar part as the "time" part and the vector part as the "space" part. The product of two quaternions can then be expressed as:

$$q_1 \cdot q_2 = (t_1 t_2 - \vec{v_1} \cdot \vec{v_2}, t_1\vec{v_2} + \vec{v_1}t_2 + \vec{v_1} \times \vec{v_2})$$

Note that we want to preserve order even for scalars. This is necessary since quaternion multiplicatoin does not commute, and we'll have to distiguish between operators (in particular differential operators) acting to the right or to the left. When we define a new operator, we'll also want to have both a right version and a left version.

4 Revisiting Maxwell's theory with quaternions

- We'll be working in quaternion space, where a 'quaternion' position will be of the form $r = ct + \vec{v} = ct + x\mathbf{i} + y\mathbf{k} + z\mathbf{k}$ where c is the speed of light.
- For an operator L, we shall denote the action direction by an arrow $\rightarrow$. For example, L acting to the left on q would be denoted by $q \leftarrow L$ and on the right by $L \rightarrow q$.
- For convenience, we define the symmetric and the antisymmetric action of an operator L:

$$\{L, q\} = \frac{1}{2}(L \rightarrow q + q \leftarrow L)$$
$$[L, q] = \frac{1}{2}(L \rightarrow q - q \leftarrow L)$$

- In analogy to the 4-potential in special relativity, let's define the electromagnetic potential as the quaternion $A = \phi + \vec{A}$.
- Similarly, we define the gradient differential operator for quaternions $\frac{d}{dr} = \frac{1}{c}\partial_t + \vec{\nabla}$.
- Let's look at the left action and the right action of $\frac{d}{dr}$ on the vector potential A:

$$\frac{d}{dr} \rightarrow A = \frac{1}{c}\partial_t\phi - \nabla \cdot \vec{A} + \frac{1}{c}\partial_t\vec{A} + \nabla\phi + \nabla \times \vec{A}$$
$$\frac{d}{dr} \leftarrow A = \frac{1}{c}\partial_t\phi - \nabla \cdot \vec{A} + \frac{1}{c}\partial_t\vec{A} + \nabla\phi - \nabla \times \vec{A}$$

- By inspection, one can define the electric and magnetic field quaternions E and B by:

$$E = -\frac{1}{2}(\frac{d}{dr} \rightarrow A + A \leftarrow \frac{d}{dr}) = -\{\frac{d}{dr}, A\}$$
$$= \underbrace{-\frac{1}{c}\partial_t\phi + \nabla \cdot \vec{A}}_{\text{extra}} - \nabla\phi - \frac{1}{c}\partial_t\vec{A}$$
$$B = \frac{1}{2}(\frac{d}{dr} \rightarrow A - A \leftarrow \frac{d}{dr}) = [\frac{d}{dr}, A]$$
$$= \nabla \times \vec{A}$$

- By inspection, we attempt to have a guess at the Maxwell field equations:

$$[\frac{d}{dr}, B] - \{\frac{d}{dr}, E\} = 0 \implies \nabla \times \vec{B} - \frac{1}{c}\partial_t T + \nabla \cdot \vec{E} - \frac{1}{c}\partial_t\vec{E} - \nabla T = 0 \quad (6)$$
$$[\frac{d}{dr}, E] + \{\frac{d}{dr}, B\} = 0 \implies \nabla \times \vec{E} + \nabla \cdot \vec{B} + \frac{1}{c}\partial_t\vec{B} = 0 \quad (7)$$

  Equating the scalar part and the vector parts of the two equations give:

$$\nabla \times \vec{B} = \frac{1}{c}\partial_t\vec{E} + \nabla T$$
$$\nabla \cdot \vec{E} = \frac{1}{c}\partial_t T$$
$$\nabla \times \vec{E} = -\frac{1}{c}\partial_t\vec{B}$$
$$\nabla \cdot \vec{B} = 0$$

  which are indeed the Maxwell equations if we identify $\frac{1}{c}\partial_t T = 4\pi\rho$ and $\frac{4\pi\vec{J}}{c} = \nabla T$.

4.1 A missing piece recovered: the Temporal Field, and its connection to heat

- The work done by a charged particle with charge e moving a distance $d\vec{x}$ in an electric field is $e\vec{E} \cdot d\vec{x}$. By the quaternion axiom however, the line element is $dr = cdt + \mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz$, so the work done by the quaternionic electromagnetic field E is $-eTcdt + e\vec{E} \cdot d\vec{x}$. One gets an additional part for the temporal field $-Tcdt$.
  - First, we note the negative sign, so it's not work *done* but energy being *absorbed* from the system for $T > 0$. For $T < 0$, energy would be added to the system.
  - Second, we note that since the differential is dt, the energy is being absorbed as the charged particle gets displaced in time rather than in space.
- We conclude that since the contribution from the temporal field T is not mechanical work, the energy that's being exchanged between the charged particle and the system must be of the form of *heat*.
- So for $T > 0$, the positively charged particles appear "cold", whereas the negatively charged particles appear "hot". This can be identified with the peltier and thomson effects of thermo-electricity!
- The temporal field T can thus be thought of as *the total heat energy per unit charge evolving per unit time.*

4.2 Emerging thermoelectric effects: a unified theory of electrothermomagnetism

- Assume a quaternion charge density $J = \rho + \frac{1}{c}\vec{J}$, where ρ is the electric charge density and $\vec{J}$ is the current density. If one now adds a source charge term to the first maxwell equation 6:

$$[\frac{d}{dr}, B] - \{\frac{d}{dr}, E\} = 4\pi J$$

Then one gets:

$$\nabla \times \vec{B} = \frac{1}{c}\partial_t \vec{E} + \nabla T + \frac{4\pi}{c}\vec{J} \tag{8}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\partial_t \vec{B} \tag{9}$$

$$\nabla \cdot \vec{E} = \frac{1}{c}\partial_t T + 4\pi\rho \tag{10}$$

$$\nabla \cdot \vec{B} = 0 \tag{11}$$

We distinguish in equation 10 the electric field arising from the temporal source $\frac{1}{c}\partial_t T$ from a free electric charge $4\pi\rho$. In the absence of a free charge, we identify the electric field from the temporal source:

$$\nabla \cdot \vec{E}_T = \frac{1}{c}\partial_t T$$

Then we can rewrite equation 10 as:

$$\nabla \cdot (\vec{E} - \vec{E}_T) = 4\pi\rho$$

where $\vec{E}$ by itself can be identified as the "driving" field corresponding to the driving e.m.f (that responsible for moving charges in wire), and $\vec{E} - \vec{E}_T$ can be identified as the "working" field responsible for the *working e.m.f* (that determining the net energy in the system).

4.2.1 Seebeck effect

- By a symmetric argument, since a temporal field results in temperature effects, then temperature K should affect the temporal field T. So $T(K)$, and we can express the temporal source using the chain rule as:

$$\frac{1}{c}\partial_t T = \frac{1}{c}\frac{dT}{dK}\partial_t K$$

where $\frac{dT}{dK}$ can be thought of as a characteristic of the medium, namely some sort of "heat capacity" of the material, and that the electric field $\vec{E}_T$ is mainly produced corresponding to a change in temperature.

- The state of a material of temperature gradient evolves according to the heat equation, effectively causing a change of temperature and thus generation of electricity as expected (the *Seebeck effect*).

4.2.2 Thomson effect

- Multiplying both sides of equation 8 by $\frac{c}{4\pi}\vec{J}\cdot\sigma^{-1}$, where σ^{-1} is the inverse of the conductivity tensor (the resistivity tensor), yields the balance equation:

$$\underbrace{\vec{J}\cdot\sigma^{-1}\vec{J}}_{\text{Joule's heat}} + \underbrace{\frac{c}{4\pi}\frac{dT}{dK}\vec{J}\cdot\sigma^{-1}(\nabla\cdot K)}_{\text{Thomson heat}} - \frac{c}{4\pi}\vec{J}\cdot\sigma^{-1}(\nabla\times\vec{B}) + \frac{1}{4\pi}\vec{J}\cdot\sigma^{-1}\partial_t\vec{E} = 0$$

  The first term is identified with Joule's irreversible heat, and the second term is identified with Thomson heat as a result of current moving in a temperature gradient. The last two terms should cancel out with the first two so that the total net energy of the isolated system is zero.

- Let's design an experiment so that the electric field $\vec{E}$ is constant in time so that the third term vanish. Then if the circulation of the magnetic field is perpendicular to the flow of current (which can be achieved with the "U" arrangement as seen in figure 1), then the fourth term vanishes. Finally, with the introduction of a heat reservoir that contributes with power $\frac{dQ}{dt}$, we will have achieved exactly the conditions of Thomson effect, and the balance equation becomes:

$$\frac{dQ}{dt} = \vec{J}\cdot\sigma^{-1}\vec{J} + \frac{c}{4\pi}\frac{dT}{dK}\vec{J}\cdot\sigma^{-1}(\nabla\cdot K) \tag{12}$$

- In the case the conductivity is a scalar (material is isotropic), one can identify equation 12 with equation .

$$\frac{dQ}{dt} = \frac{\vec{J}^2}{\sigma} - \mathcal{K}\vec{J}\cdot\nabla K$$

  where $\mathcal{K} = -\frac{c}{4\pi\sigma}\frac{dT}{dK}$ is the thomson coeffecient with the addition to contribution from Joule's heat. From that, one can interpret the "heat capacity" characteristic parameter of the medium $\frac{dT}{dK}$ to be the product of the Thomson coeffecient and the conductivity.

4.2.3 Maxwell Equations in Matter

- Then in absence of free currents (when $\vec{J}=0$), similarly to how $\vec{E}_T$ was defined, one defines the magnetic field due to the temporal field $\nabla\times\vec{B}_T = \frac{1}{c}\partial_t\vec{E} + \nabla T$. Then one can identify the displacement field with $\vec{D} = \vec{E} - \vec{E}_T$ and the magnetisation field by $\vec{H} = \vec{B} - \vec{B}_T$ and argue that the macroscopic description is packaged inside the temporal field expressions.